Correlations and symmetry breaking in gapped matrix models

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Some puzzles which arise in matrix models with multiple cuts are presented. They are present in the smoothed eigenvalue correlators of these models. First a method is described to calculate smoothed eigenvalue correlators in random matrix models with eigenvalues distributed in a single cut. Previous known results are reproduced. The method is extended to symmetric two-cut random matrix models. The correlators are written in a form suitable for application to mesoscopic systems. Connections are made with the smooth correlators derived using the orthogonal polynomial method. A few interesting observations are made regarding even and odd density-density correlators and crossover correlators in Z_2 symmetric random matrix models. A symmetry breaking parameter C is identified in the smooth correlators for all $\beta=1$, 2, and 4. [S1063-651X(99)02504-0]

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I. INTRODUCTION

Matrix models have been used in a wide variety of applications, starting from quantum chaotic systems to condensed matter, quantum chromodynamics, and string theory. The recent period has seen a large increase in our understanding of the properties of these models. In this work we have been interested in highlighting some unusual properties of two-cut random matrix models that have arisen in our study. The results are unexpected as they are not seen in matrix models when the density of eigenvalues has a connected support. Indeed there it is well known [1,2] that the correlator is universal, i.e., independent of the specific potential V which defines the probability measure. This is the basis for the theory under the universality of conductance fluctuations in mesoscopic systems [3]. At first sight one is tempted to think that this universality persists when the potential is such that the support splits into disconnected segments. But it is found that, if indeed it is again universal, it belongs to a different universality class. If the standard large-N limit (the random matrices are $N\times N$) yields the smoothed correlation functions up here to an arbitrary constant, different methods report different results for this constant. Furthermore, there are differences between these correlators when the size N of the matrices is an even or an odd integer. It is a rather intriguing phenomenon and, for instance, it is not clear how the naive renormalization-group approach [4] which consisted of integrating out one line and one row could deal with such situations. We attempt here to understand and to give a unified picture of these results.

The paper is divided as follows. It starts by establishing the notation and conventions and describes completely the method used for the model with a single-cut density of eigenvalues. Previously known results are reproduced [1–3]. Then the method is extended to the model with two cuts in the density of eigenvalues, restricted to symmetric potentials. Afterwards we develop the formalism to include asymmetric potentials. Here an arbitrariness remains as the constraint on the filling factor of the two parts of the support is not fixed at leading order in the large-N limit. The large-N equations for the correlator leave us with an undetermined constant C. Previous methods using the orthogonal polynomials and loop equations give different results for this constant. The orthogonal polynomial method is briefly outlined and the resulting correlators are sensitive to the even and oddness of the number of eigenvalues. Further, the constant C is different for the even and odd correlators found by the orthogonal polynomial method and that found by the loop equations. Thus C is identified as a symmetry breaking parameter. The new generalized technique described here allows extending the results for the smoothed two-point density-density correlator to all $\beta=1,2$, and 4. The conclusion summarizes these new results and attempts to give an explanation of these puzzles.

II. NOTATION, CONVENTIONS

We establish the notations and conventions and develop a method, which we extend to the two-cut model, to derive eigenvalue correlators for random matrix models with a single-cut density of eigenvalues.

Let us work with an ensemble of random $N\times N$ matrices, with a probability distribution

$$P(M)=\frac{1}{Z}\exp(-N\,\mathrm{Tr}\,V(M)). \tag{2.1}$$

Define the operator for the density of eigenvalues

$$\rho(x)=\frac{1}{N}\sum_{i=1}^{N}\delta(x-\lambda_i) \tag{2.2}$$

and

$$\overline{\rho}(x)=\langle\rho(x)\rangle=\int P(M)\frac{1}{N}\mathrm{Tr}\,\delta(x-M)[dM], \tag{2.3}$$

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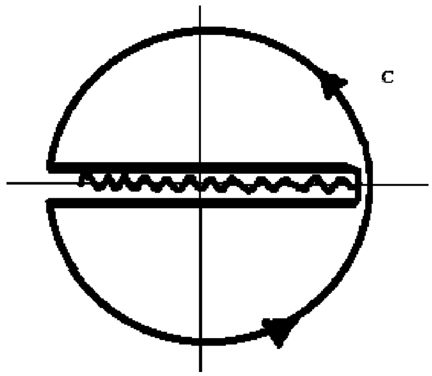

FIG. 1. The complex z plane with one cut and contour used for evaluating the two-point density-density correlator for the one-cut random matrix model.

in which $[dM]$ is the invariant measure on the three standard ensembles [5] $\beta=1,2,4$, respectively, invariant under the orthogonal, unitary, or symplectic groups. Since $P(M)$ gives a factor $\exp(-N^2\int V(x)\rho(x)dx)$, we have

$$\frac{\delta\overline{\rho}(x)}{\delta V(y)}=-N^2\langle\rho(x)\rho(y)\rangle_c. \tag{2.4}$$

In the large-N limit, we know that

$$P\int_a^b\frac{\overline{\rho}(y)}{x-y}dy=\frac{1}{\beta}V'(x). \tag{2.5}$$

The solution is found through the averaged resolvent

$$G(z)=\left\langle\frac{1}{N}\mathrm{Tr}\frac{1}{z-M}\right\rangle=\int_a^b\frac{\overline{\rho}(y)}{z-y}dy$$
$$=\frac{1}{\beta}V'(z)-P(z)\sqrt{(z-a)(z-b)}. \tag{2.6}$$

Then

$$\pi\overline{\rho}(\lambda)=P(\lambda)\sqrt{(\lambda-a)(b-\lambda)}. \tag{2.7}$$

We then need to express P as a functional of V. There are various equivalent expressions, and in the following several will be needed. We follow here [3] and begin by multiplying Eq. (2.6) by $\sqrt{(z-a)(z-b)}/(z-u)$ and integrate z over a large circle C, see Fig. 1.

Since $G(z)\approx 1/z$ at infinity,

$$\text{(i)}\quad\oint_c\frac{G(z)\sqrt{(z-a)(z-b)}}{z-u}\frac{dz}{2i\pi}=1, \tag{2.8}$$

$$\text{(ii)}\quad\oint_c\frac{P(z)[\sqrt{(z-a)(z-b)}]^2}{z-u}\frac{dz}{2i\pi}=P(u)(u-a)(u-b), \tag{2.9}$$

$$\text{(iii)}\quad\oint_c\frac{V'(z)\sqrt{(z-a)(z-b)}}{z-u}\frac{dz}{2i\pi}$$
$$=V'(u)\sqrt{(u-a)(u-b)}$$
$$-\frac{1}{\pi}\int_a^b dx\frac{V'(x)\sqrt{(x-a)(b-x)}}{x-u}. \tag{2.10}$$

Therefore we obtain from Eq. (2.6)

$$1=\frac{1}{\beta}V'(u)\sqrt{(u-a)(u-b)}$$
$$-\frac{1}{\pi\beta}\int_a^b dx\frac{V'(x)\sqrt{(x-a)(b-x)}}{x-u}$$
$$+P(u)(u-a)(b-u). \tag{2.11}$$

Let u approach the real axis on the cut. The integral in Eq. (2.11) has an imaginary part which cancels the first term of the right-hand side, which is pure imaginary. The real part of Eq. (2.11) gives

$$P(\lambda)(\lambda-a)(b-\lambda)$$
$$=1+\frac{1}{\pi\beta}P\int_a^b dx\frac{V'(x)\sqrt{(x-a)(b-x)}}{x-\lambda}, \tag{2.12}$$

i.e.,

$$\overline{\rho}(\lambda)=\frac{1}{\pi}\frac{1}{\sqrt{(\lambda-a)(b-\lambda)}}$$
$$\times\left(1+\frac{1}{\pi\beta}P\int_a^b dx\frac{V'(x)\sqrt{(x-a)(b-x)}}{x-\lambda}\right). \tag{2.13}$$

Now we vary V. Let us first ignore the variation of a and b (it is proved to be right below). Then

$$\frac{\delta\overline{\rho}(\lambda)}{\delta V(\mu)}=\left(\frac{\partial\overline{\rho}(\lambda)}{\partial V(\mu)}\right)_{a,b}+\left(\frac{\partial\overline{\rho}(\lambda)}{\partial a}\right)_{V,b}\frac{\delta a}{\delta V(\mu)}$$
$$+\left(\frac{\partial\overline{\rho}(\lambda)}{\partial b}\right)_{V,a}\frac{\delta b}{\delta V(\mu)} \tag{2.14}$$

on the right-hand side (r.h.s.). $\overline{\rho}$ is being treated as a function of V,a,b as given on the r.h.s. of Eq. (2.13). We show later that $(\partial\overline{\rho}/\partial a)_{V,b}=0$. $[(\partial\overline{\rho}/\partial a)_{V,b}$ is of course not the total derivative of $\overline{\rho}$ with respect to a.] Then

$$\frac{\delta\overline{\rho}(\lambda)}{\delta V(\mu)}=\frac{1}{\pi^2\beta}\frac{1}{\sqrt{(\lambda-a)(b-\lambda)}}\frac{\partial}{\partial\mu}\frac{\sqrt{(\mu-a)(b-\mu)}}{\lambda-\mu} \tag{2.15}$$

and one verifies easily that the result is symmetric under exchange of λ and μ as it should be. Note that the potential V has disappeared from the correlator, except indirectly through the end points a and b of the cut. This universality follows here trivially from the linearity of the (ρ,V) relation. The fact that, apart from a normalization, the result is independent of β was also expected: indeed in a Feynman graph representation, the ensembles differ by the orientability of the surfaces built with the diagrams. The large-N limit is given by planar diagrams, which are orientable, and it is only at the level of $1/N$ corrections that the differences between these ensembles would appear; for the unitary ensemble for instance, the corrections to Eq. (2.15) would vanish at order $1/N$, and not for the other values of β.

Terms of the type

$$\left(\frac{\delta\overline{\rho}}{\delta a}\right)_{V,b}\frac{\delta a}{\delta V(\mu)} \tag{2.16}$$

have been ignored. The claim is that they vanish, but that is the only (slightly) tricky part. The representation Eq. (2.13) is appropriate, among several other possibilities, because if one differentiates inside the integral with respect to a, it is still a meaningful integral. So let us calculate

$$\left(\frac{\partial\overline{\rho}}{\partial a}\right)_{V,b}=-\frac{1}{2}\frac{1}{(\lambda-a)}\overline{\rho}(\lambda)+\frac{1}{2\pi^2\beta}\frac{1}{\sqrt{(\lambda-a)(b-\lambda)}}\times P\int_a^b V'(x)\sqrt{\frac{b-x}{x-a}}\frac{1}{(x-\lambda)}dx. \tag{2.17}$$

To prove that this is zero, let us return to Eq. (2.6) and multiply it by $\sqrt{(z-b)/(z-a)}[1/(z-u)]$ and integrate again over a circle of large radius. Then

$$\oint_c G(z)\sqrt{\frac{z-b}{z-a}}\frac{1}{z-u}dz=\oint_c\frac{dz}{z^2}=0 \tag{2.18}$$

as for large $z, G(z)=1/z, \sqrt{z-b/z-a}\approx 1$, and $1/(z-u)\approx 1/z$, while the second and third terms become

$$-\oint_c\frac{P(z)(z-b)}{(z-u)}\frac{dz}{2i\pi}=-P(u)(u-b) \tag{2.19}$$

and

$$\frac{1}{2}\oint_c\sqrt{\frac{z-b}{z-a}}\frac{1}{z-u}V'(z)\frac{dz}{2i\pi}=\frac{1}{2}\sqrt{\frac{u-b}{u-a}}V'(u)-\frac{1}{2\pi}\int_a^b\sqrt{\frac{b-x}{x-a}}\frac{V'(x)}{x-u}dx. \tag{2.20}$$

Taking $u=\lambda+i\epsilon$ and using $1/\alpha-i\epsilon=P/\alpha+i\pi\delta(\alpha)$, the integral in Eq. (2.20) has a part which cancels the first term leaving

$$\oint_c\sqrt{\frac{z-b}{z-a}}\frac{1}{z-u}V'(z)\frac{dz}{2i\pi}=-\frac{1}{\pi}P\int_a^b\sqrt{\frac{b-x}{x-a}}\frac{V'(x)}{x-u}dx. \tag{2.21}$$

Repeating all the steps which led to Eq. (2.15), i.e., combining Eq. (2.18), Eq. (2.19), and Eq. (2.21), one finds an expression for $\overline{\rho}$ from

$$\frac{1}{\pi\beta}P\int_a^b\sqrt{\frac{b-x}{x-a}}\frac{V'(x)}{x-\lambda}dx=P(\lambda)(b-\lambda), \tag{2.22}$$

which is

$$\frac{\overline{\rho}(\lambda)}{(\lambda-a)}=\frac{1}{\pi^2\beta\sqrt{(\lambda-a)(b-\lambda)}}\times P\int_a^b\sqrt{\frac{b-x}{x-a}}\frac{1}{(x-\lambda)}V'(x)dx \tag{2.23}$$

thus proving that $(\partial\overline{\rho}/\partial a)_{V,b}=0$. This completes the proof for the single-cut correlator.

III. THE DOUBLE WELL

Now let us extend the result to eigenvalues distributed in two disjoint bands ($[-b,-a]\cup[a,b]$). Let us first restrict ourselves to even potentials, i.e.,

$$P(M)=Z^{-1}\exp\ (-N\,\mathrm{Tr}\,V(M)),\quad P(-M)=P(M), \tag{3.1}$$

which implies for the resolvent

$$G(-z)=-G(z). \tag{3.2}$$

Since we restrict ourselves to even potentials, we cannot take a functional derivative of $\rho(\lambda)$ with respect to an arbitrary $V(\mu)$, but we can fold the integrations over the positive part of the spectrum and then vary the potential. Now

$$\mathrm{Tr}\,V(M)=N\int_{-\infty}^{+\infty}d\lambda\,\rho(\lambda)V(\lambda)=N\int_0^{\infty}d\lambda\,V(\lambda)[\rho(\lambda)+\rho(-\lambda)]. \tag{3.3}$$

Consequently,

$$-\frac{1}{N^2}\frac{\delta\overline{\rho}(\lambda)}{\delta V(\mu)}=\langle\rho(\lambda)\rho(\mu)\rangle_c+\langle\rho(\lambda)\rho(-\mu)\rangle_c, \tag{3.4}$$

where use has been made of

$$\frac{\delta V'(x)}{\delta V(\mu)}=\delta'(x-\mu). \tag{3.5}$$

In the large-N limit again

$$G(z)=\frac{1}{\beta}V'(z)-P(z)\sqrt{\sigma(z)} \tag{3.6}$$

with $\sigma(z)\equiv(z^2-a^2)(z^2-b^2)$. Note that this equation determines uniquely $P(z), a$, and b; indeed take

$$\deg V=2n,\quad\rightarrow\deg[P]=2n-3;$$

$P(z)$ has to be odd,

$$P(z)=\alpha_1 z+\alpha_2 z^3+\cdots+\alpha_{n-1}z^{2n-3}. \tag{3.7}$$

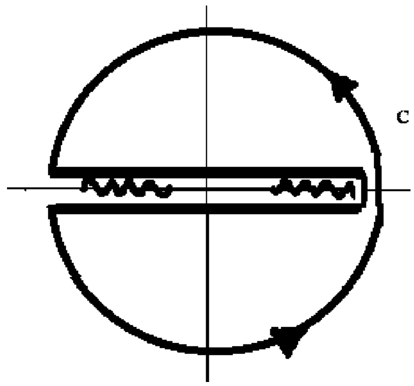

FIG. 2. The complex z plane with two cuts and contour used for evaluating the two-point density-density correlator for the two-cut random matrix model.

We thus have $(n-1)+2$ unknowns. Since $G(z)\approx_{z\to\infty}1/z$ we have to fix the coefficients of Eq. (3.6) at infinity from $z^{2n-1},z^{2n-3},\dots,z^{1},z^{-1}\to(n+1)$ conditions. Therefore no ''filling'' parameter creeps into the problem (although the question of spontaneous symmetry breaking may be eliminated by the assumptions here).

Now we take Eq. (3.6), multiply by $\sqrt{\sigma(z)/z(z-u)}$, and integrate over a large circle in the z plane [using $\sqrt{\sigma(z)/z^2-u^2}$ also has been checked to give the same equation], see Fig. 2. We obtain

$$1=\frac{V'(u)\sqrt{\sigma(u)}}{\beta u}-\frac{2i}{2i\pi\beta}\int_b^a\frac{V'(x)\sqrt{|\sigma(x)|}}{x(x-u)}+\frac{2i}{2i\pi\beta}\int_{-a}^{-b}\frac{V'(x)\sqrt{|\sigma(x)|}}{x(x-u)}-\frac{P(u)\sigma(u)}{u}\tag{3.8}$$

which simplifies to

$$1+\frac{P(u)\sigma(u)}{u}=\frac{V'(u)\sqrt{|\sigma(u)|}}{\beta u}-\frac{1}{\pi\beta}\int_b^a\frac{V'(x)\sqrt{|\sigma(x)|}}{x}\times\left(\frac{1}{(x-u)}+\frac{1}{(x+u)}\right).\tag{3.9}$$

We now let $u=\lambda+i\epsilon$ approach the cut, say the one on the right (it does not matter),

$$1+\frac{P(\lambda)\sigma(\lambda)}{\lambda}=\frac{V'(\lambda)\sqrt{|\sigma(\lambda)|}}{\beta\lambda}-\frac{1}{\pi\beta}\int_b^a\frac{V'(x)\sqrt{|\sigma(x)|}}{x}\frac{1}{(x+\lambda)}-\frac{1}{\pi\beta}\int_b^a\frac{V'(x)\sqrt{|\sigma(x)|}}{x}\frac{1}{(x-\lambda-i\epsilon)}.\tag{3.10}$$

In the last integral use $1/\alpha-i\epsilon=PP/\alpha+i\pi\delta(\alpha)$ and we obtain

$$1+\frac{P(\lambda)\sigma(\lambda)}{\lambda}=-\frac{1}{\pi\beta}P\int_b^a\frac{V'(x)\sqrt{|\sigma(x)|}}{x}\times\left(\frac{1}{(x-\lambda)}+\frac{1}{(x+\lambda)}\right)dx.\tag{3.11}$$

Let us take the derivative with respect to $V(\mu)$(μ is >0 by definition of V),

$$\frac{\sigma(\lambda)}{\lambda}\frac{\delta P(\lambda)}{\delta V(\mu)}=\frac{1}{\pi\beta}\frac{\partial}{\partial\mu}\frac{\sqrt{|\sigma(\mu)|}}{\mu}\left(\frac{1}{(\mu-\lambda)}+\frac{1}{(\mu+\lambda)}\right)\tag{3.12}$$

[assuming that we can show here as usual that a counterpart of $(\delta\bar{\rho}/\delta a)_{V,b}[\delta a/\delta V(\mu)]$ and $(\delta\bar{\rho}/\delta b)_{V,a}[\delta b/\delta V(\mu)]$ vanishes, see Appendix A for a proof, the exact same steps can be followed here]. Then

$$\bar{\rho}(\lambda)=\frac{1}{2\pi}\sqrt{|\sigma(\lambda)|}P(\lambda)\ (\lambda>0),\tag{3.13}$$

$$\frac{\delta\bar{\rho}(\lambda)}{\delta V(\mu)}=2\frac{1}{2\pi\beta}\frac{\sqrt{|\sigma(\lambda)|}}{\sigma(\lambda)}\lambda\frac{\partial}{\partial\mu}\frac{\sqrt{(\mu^2-b^2)(a^2-\mu^2)}}{\mu^2-\lambda^2}=-\frac{1}{\pi\beta}\frac{\lambda\mu}{\sqrt{|\sigma(\lambda)||\sigma(\mu)|}}\frac{1}{(\mu^2-\lambda^2)^2}\times[2\lambda^2\mu^2-(\lambda^2+\mu^2)(a^2+b^2)+2a^2b^2].\tag{3.14}$$

Let us check immediately the $b=0$ limit

$$\frac{\lambda}{\sqrt{|\sigma(\lambda)|}}\to\frac{1}{\sqrt{(a^2-\lambda^2)}}\tag{3.15}$$

and the rest looks unfamiliar; but if we remember that we are computing

$$\rho_2(\lambda,\mu)+\rho_2(\lambda,-\mu)\tag{3.16}$$

and

$$\frac{a^2-\lambda\mu}{(\lambda-\mu)^2}+\frac{a^2+\lambda\mu}{(\lambda+\mu)^2}=-2\frac{[2\lambda^2\mu^2-a^2(\lambda^2+\mu^2)]}{(\lambda^2-\mu^2)^2},\tag{3.17}$$

we check that this result agrees as expected for $b=0$ with the single-cut result. Therefore for a symmetric double well, assuming no spontaneous symmetry breaking, we have the undisputable answer for $\rho_2(\lambda,\mu)+\rho_2(\lambda,-\mu)$, i.e., Eq. (3.14). Note that, as expected, the short distance behavior of $\rho_2(\lambda,\mu)$ is the same as for the single well with only one cut.

IV. ASYMMETRIC DOUBLE WELL

In order to extract $\rho_2(\lambda,\mu)$ alone we have to consider arbitrary potentials instead of restricting ourselves to even (Z_2) symmetric potentials as we have done in the above section.

Again let us start with

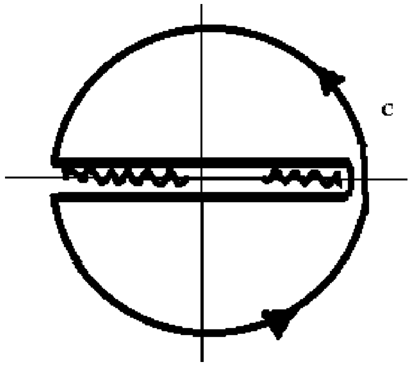

FIG. 3. The complex z plane with asymmetric two cuts and contour used for evaluating the two-point density-density correlator for the asymmetric two-cut random matrix model.

$$\rho(\lambda)=\frac{1}{N}\sum_{i=1}^{N}\delta(\lambda-\lambda_i) \tag{4.1}$$

with

$$\overline{\rho(\lambda)}=\langle\rho(\lambda)\rangle. \tag{4.2}$$

Since the weight contains

$$\exp(-N\,\mathrm{Tr}\,V)=\exp\left(-N^2\int V(\lambda)\rho(\lambda)d\lambda\right), \tag{4.3}$$

$$\rho_2^c(\lambda,\mu)=\langle\rho(\lambda)\rho(\mu)\rangle_c=-\frac{1}{N^2}\frac{\delta\overline{\rho(\lambda)}}{\delta V(\mu)}, \tag{4.4}$$

$$\overline{\rho(\lambda)}=-\frac{1}{\pi}\,\mathrm{Im}\,G(\lambda+i0), \tag{4.5}$$

$$G(z)=\left\langle\frac{1}{N}\mathrm{Tr}\frac{1}{z-M}\right\rangle, \tag{4.6}$$

$$G(z)=\frac{1}{\beta}V'(z)-P(z)\sqrt{\sigma(z)}, \tag{4.7}$$

$$\sigma(z)=\Pi_{i=1}^{4}(z-a_i). \tag{4.8}$$

See Fig. 3.

The support of the eigenvalues consists of the two segments $[a_1,a_2]$ and $[a_3,a_4]$, (we assume that they are labeled by increasing order); the positivity of $\rho(\lambda)$ is satisfied provided the polynomial $P(z)$ has an odd number of zeros between a_2 and a_3. Contrary to the two previous cases, Eq. (4.7) is not sufficient to determine fully the polynomial $P(z)$ and the four end points of the two cuts. Counting parameters and equations one sees readily that we miss one parameter, which we can take as the filling factor of one of the two wells. This factor remains undetermined at this level of the large-N limit, and we would have to return to a minimization of the free energy to fix it. However, since this parameter is not fixed at this leading order, we may ignore it and proceed as before for finding the leading order of the correlator.

We denote

$$\begin{aligned}\epsilon_\lambda&=+1,\ a_3<\lambda<a_4,\\ &=-1,\ a_1<\lambda<a_2.\end{aligned} \tag{4.9}$$

Then

$$\sqrt{\sigma(\lambda\pm i0)}=\pm i\epsilon_\lambda\sqrt{|\sigma(\lambda)|}. \tag{4.10}$$

Therefore

$$2\pi\rho(\lambda)=\epsilon_\lambda\sqrt{|\sigma(\lambda)|}P(\lambda) \tag{4.11}$$

and positivity implies that $\mathrm{sgn}P(\lambda)=\epsilon_\lambda$. By multiplication by $\sqrt{\sigma(z)/(z-u)(z-v)}$ integration over a large circle, we obtain

$$\begin{aligned}1+&\frac{P(u)\sigma(u)-P(v)\sigma(v)}{(u-v)}\\ &=\frac{1}{\beta}\frac{V'(u)\sqrt{\sigma(u)}-V'(v)\sqrt{\sigma(v)}}{u-v}\\ &\quad-\frac{1}{\pi\beta}\left(\int_{a_3}^{a_4}-\int_{a_1}^{a_2}\right)\frac{dx\,V'(x)\sqrt{|\sigma(x)|}}{(x-u)(x-v)}.\end{aligned} \tag{4.12}$$

Let $u=\lambda+i\epsilon$ and $v=\mu+i\eta$. Then

$$\begin{aligned}1+&\frac{P(\lambda)\sigma(\lambda)-P(\mu)\sigma(\mu)}{\lambda-\mu}\\ &=\frac{1}{\pi\beta}P\left(\int_{a_1}^{a_2}-\int_{a_3}^{a_4}\right)\frac{V'(x)\sqrt{|\sigma(x)|}}{(x-\lambda)(x-\mu)},\end{aligned} \tag{4.13}$$

$$R(\lambda,\nu)=\frac{\delta P(\lambda)}{\delta V(\nu)}, \tag{4.14}$$

$$\frac{\sigma(\lambda)R(\lambda,\nu)-\sigma(\mu)R(\mu,\nu)}{\lambda-\mu}=\epsilon_\nu\frac{1}{\beta\pi}\frac{\partial}{\partial\nu}\frac{\sqrt{|\sigma(\nu)|}}{(\nu-\lambda)(\nu-\mu)} \tag{4.15}$$

[assuming once more an equivalent form of $(\delta\overline{\rho}/\delta a_i)(\delta a_i/\delta V)$ to be zero; see Appendix A]. Hence

$$\begin{aligned}\sigma(\lambda)R(\lambda,\nu)-\frac{\epsilon_\nu}{\pi\beta}\frac{\partial}{\partial\nu}\frac{\sqrt{|\sigma(\nu)|}}{\nu-\lambda}&=\sigma(\mu)R(\mu,\nu)\\ &\quad-\frac{\epsilon_\nu}{\pi\beta}\frac{\partial}{\partial\nu}\frac{\sqrt{|\sigma(\nu)|}}{\nu-\mu}.\end{aligned} \tag{4.16}$$

From these equations one finds

$$\sigma(\lambda)R(\lambda,\nu)-\frac{\epsilon_\nu}{\pi\beta}\frac{\partial}{\partial\nu}\frac{\sqrt{|\sigma(\nu)|}}{\nu-\lambda}=\frac{h_{\epsilon_\nu}(\nu)}{\sqrt{|\sigma(\nu)|}} \tag{4.17}$$

and we are left with two unknown functions h_+ and h_- of a single variable. This gives the connected correlator

$$\rho_2^c(\lambda,\mu)=-\frac{1}{2\pi N^2}\epsilon_\lambda\sqrt{|\sigma(\lambda)|}R(\lambda,\mu), \tag{4.18}$$

i.e.,

$$2\pi N^2\rho_2^c(\lambda,\mu)=\frac{\epsilon_\lambda}{\sqrt{|\sigma(\lambda)|}}\left\{\frac{\epsilon_\mu}{\pi\beta}\frac{\partial}{\partial\mu}\frac{\sqrt{|\sigma(\mu)|}}{\mu-\lambda}+\frac{h_{\epsilon_\mu}(\mu)}{\sqrt{|\sigma(\mu)|}}\right\}. \tag{4.19}$$

From its definition the two-point correlator is symmetric under exchange of the two eigenvalues

$$\rho_2^c(\lambda,\mu)=\rho_2^c(\mu,\lambda). \tag{4.20}$$

This imposes the following constraints:

$$h_+(\mu)+h_-(\mu)=0 \tag{4.21}$$

and

$$\beta\pi[h_+(\mu)-h_+(\lambda)]=\sqrt{|\sigma(\lambda)|}\frac{\partial}{\partial\lambda}\frac{\sqrt{|\sigma(\lambda)|}}{\lambda-\mu}-\sqrt{|\sigma(\mu)|}\frac{\partial}{\partial\mu}\frac{\sqrt{|\sigma(\mu)|}}{\mu-\lambda}. \tag{4.22}$$

A straightforward algebra gives from there the function h_+ up to an arbitrary constant:

$$h_+(\lambda)=\frac{1}{\pi\beta}\left(\lambda^2-\frac{1}{2}s\lambda+C\right) \tag{4.23}$$

with

$$s=a_1+a_2+a_3+a_4. \tag{4.24}$$

We are thus left with one undetermined constant in the two-point function:

$$4\pi^2N^2\rho_2^c(\lambda,\mu)=\frac{\epsilon_\lambda\epsilon_\mu}{\beta\sqrt{|\sigma(\lambda)|}\sqrt{|\sigma(\mu)|}}\left(\frac{\sigma(\lambda)+\sigma(\mu)}{(\lambda-\mu)^2}+\frac{\sigma'(\lambda)-\sigma'(\mu)}{(\lambda-\mu)}+\lambda^2+\mu^2-\frac{s}{2}(\lambda+\mu)+2C\right). \tag{4.25}$$

Let us verify that, without any restriction on the constant C, this result satisfies the normalization condition

$$\int d\nu\rho_2^c(\lambda,\nu)=0, \tag{4.26}$$

which follows from the definition of ρ_2^c. Returning to Eq. (4.20),

$$\int d\mu\frac{\partial}{\partial\mu}\frac{\sqrt{\sigma(\mu)}}{\mu-\lambda}=\int_{a_1}^{a_2}d\mu\frac{\partial}{\partial\mu}\frac{\sqrt{\sigma(\mu)}}{\mu-\lambda}+\int_{a_3}^{a_4}d\mu\frac{\partial}{\partial\mu}\frac{\sqrt{\sigma(\mu)}}{\mu-\lambda}=0 \tag{4.27}$$

since σ vanishes at the end points. [This point is in fact slightly delicate, since there is a nonintegrable singularity at $\mu=\lambda$. In the literature concerning the application of random matrices to the calculation of the fluctuations of the conductance in mesoscopic systems [3], this integration throughout the singularity is done in a routine way. A proper justification of the procedure implies returning to the true correlation function, before the smoothing which produces spurious short-distance singularities through replacements such as $(\sin^2 x/x^2\to 1/2x^2)$. The smoothing is produced here by the large-N limit.] Next consider

$$\oint dz\frac{(z^2-sz/2)}{\sqrt{\sigma(z)}} \tag{4.28}$$

over a large circle. Note that $\sqrt{\sigma(z)}=z^2[1-s/2z+O(1/z^2)]$. Consequently $z^2-sz/2/\sqrt{\sigma(z)}=1+O(1/z^2)$. Since there is no coefficient of $1/z$, the integral vanishes. Shrinking the contour over the cuts, we obtain

$$\left(\int_{a_1}^{a_2}-\int_{a_3}^{a_4}\right)d\nu\frac{\nu^2-s\nu/2}{\sqrt{|\sigma(\nu)|}}=0. \tag{4.29}$$

Therefore

$$2\pi N^2\left(\int_{a_1}^{a_2}d\mu\rho_2^c+\int_{a_3}^{a_4}d\mu\rho_2^c\right)=\frac{C}{\pi\beta}\frac{\epsilon_\lambda}{\sqrt{|\sigma(\lambda)|}}\left(\int_{a_3}^{a_4}\frac{d\mu}{\sqrt{|\sigma(\mu)|}}-\int_{a_1}^{a_2}\frac{d\mu}{\sqrt{|\sigma(\mu)|}}\right). \tag{4.30}$$

Again taking a very large circle

$$\oint\frac{dz}{\sqrt{\sigma(z)}}=0 \tag{4.31}$$

since the coefficient of $1/z$ vanishes. Therefore, shrinking the circle,

$$\int_{a_3}^{a_4}\frac{d\mu}{\sqrt{|\sigma(\mu)|}}=\int_{a_1}^{a_2}\frac{d\mu}{\sqrt{|\sigma(\mu)|}} \tag{4.32}$$

which shows that the normalization is correct for any value of C.

Let us specialize to the symmetric double well ($a_1=-b$, $a_2=-a$, $a_3=a$, and $a_4=b$)

$$|\sigma(\lambda)|=(\lambda^2-a^2)(b^2-\lambda^2). \tag{4.33}$$

After a few lines

$$2\pi^2\rho_2^c(\lambda,\mu)=\frac{\epsilon_\lambda\epsilon_\mu}{\beta\sqrt{|\sigma(\lambda)||\sigma(\mu)|}}\frac{1}{(\mu-\lambda)^2}\times[C(\mu-\lambda)^2+\lambda\mu(\lambda\mu-a^2-b^2)+a^2b^2]. \tag{4.34}$$

We want now to make several remarks. (i) The result is manifestly symmetric under $\lambda\leftrightarrow\nu$. (ii) C remains an unknown function of a and b. (iii) If we assume that C vanishes when $b=0$, then since

$$\lim_{a\to 0}\sqrt{|\sigma(\lambda)|}=\epsilon_\lambda\lambda(a^2-\lambda^2) \tag{4.35}$$

we recover the single-band result. (iv) The cross-correlator $\rho_c^2(\lambda,-\nu)$ is simply Eq. (4.34) with ν replaced by $-\nu$. Combining Eq. (4.34) with $\rho_c^2(\lambda,-\nu)$ we reproduce the result Eq. (3.14) in Sec. III. (v) We also note that for $C=-\frac{1}{2}\{(a^2+b^2)-(a+b)^2[E(k)/K(k)]\}$ we recover the connected density-density correlator derived from the connected Green's function [where $E(k)$ and $K(k)$ are complete elliptic integrals of first and second kind and $k=2\sqrt{ab}/(a+b)$] in Eq. (15) of Ref. [6]. (vi) The result for the two-point density-density correlator has been generalized to all $\beta=1,2,4$ with the end points a_1,a_2,a_3,a_4,a,b all functions of beta and the parameters contained in the potential $V(x)$. Note that this is a nontrivial generalization as other methods, for example the orthogonal polynomial method, do not extend to $\beta=1,4$ easily as they involve skew orthogonal polynomials.

V. ORTHOGONAL POLYNOMIALS, THE KERNEL, ODD AND EVEN N

This section follows the notation of Ref. [7]. Let us calculate the kernel $K_N(\mu,\nu)$ for N even and N odd,

$$K_N(\mu,\nu)=\sqrt{\frac{R_N}{N}}\left(\frac{\psi_N(\mu)\psi_{N-1}(\nu)-\psi_{N-1}(\mu)\psi_N(\nu)}{(\mu-\nu)}\right). \tag{5.1}$$

The asymptotic ansatz for $\psi_n(\lambda)$ for $N\to\infty$ but $N-n$ finite is

$$\begin{aligned}\psi_n(\lambda)&=\frac{1}{\sqrt{f(\lambda)}}\left[\cos[N\zeta-(N-n)\phi+\chi+(-1)^n\eta](\lambda)+O\left(\frac{1}{N}\right)\right],\\ f(\lambda)&=\frac{\pi}{2\lambda}\left(\frac{b^2-a^2}{2}\right)\sin 2\phi(\lambda),\\ \zeta'(\lambda)&=-\pi\rho(\lambda),\\ \cos 2\phi(\lambda)&=\frac{\lambda^2-(a^2+b^2)/2}{(b^2-a^2)/2},\\ \cos 2\eta(\lambda)&=\frac{b\cos\phi(\lambda)}{\lambda},\\ \sin 2\eta(\lambda)&=\frac{a\sin\phi(\lambda)}{\lambda}.\end{aligned} \tag{5.2}$$

Substituting

$$\psi_N(\lambda)=\frac{1}{\sqrt{f(\lambda)}}\{\cos[N\zeta-(N-N)\phi+\chi+(-1)^N\eta](\lambda)\}, \tag{5.3}$$

$$\psi_{N-1}(\lambda)=\frac{1}{\sqrt{f(\lambda)}}\{\cos[N\zeta-(N-N+1)\phi+\chi+\times(-1)^{(N-1)}\eta](\lambda)\}, \tag{5.4}$$

we get

$$\begin{aligned}K_N(\mu,\nu)=&\frac{\sqrt{R_N}}{N(\mu-\nu)\sqrt{f(\mu)f(\nu)}}\cos Nh(\mu)\cos Nh(\nu)[\cos\phi(\nu)\cos 2\eta(\nu)-(-1)^N\sin\phi(\nu)\sin 2\eta(\nu)\\ &-\cos\phi(\mu)\cos 2\eta(\mu)+(-1)^N\sin\phi(\mu)\sin 2\eta(\mu)]+\sin Nh(\nu)\cos Nh(\mu)[\sin\phi(\nu)\cos 2\eta(\nu)\\ &+(-1)^N\sin 2\eta(\nu)\cos\phi(\nu)]-\sin Nh(\mu)\cos Nh(\nu)[\sin\phi(\mu)\cos 2\eta(\mu)+(-1)^N\sin 2\eta(\mu)\cos\phi(\mu)],\end{aligned} \tag{5.5}$$

where

$$Nh(\mu)=[N\zeta+\chi+(-1)^N\eta](\mu). \tag{5.6}$$

On simplifying further

$$K_N(\mu,\nu)=\frac{\sqrt{R_N}}{2N(\mu-\nu)\sqrt{f(\mu)f(\nu)}}\Bigg[\{\cos[Nh(\mu)+Nh(\nu)]+\cos[Nh(\mu)-Nh(\nu)]\}$$
$$\times\left(\frac{b}{\nu}\cos^2\phi(\nu)-(-1)^N\frac{a}{\nu}\sin^2\phi(\nu)-\frac{b}{\mu}\cos^2\phi(\mu)+(-1)^N\frac{a}{\mu}\sin^2\phi(\mu)\right)$$
$$+\{\sin[Nh(\nu)+Nh(\mu)]+\sin[Nh(\nu)-Nh(\mu)]\}\left(\frac{b}{\nu}+(-1)^N\frac{a}{\nu}\right)\sin\phi(\nu)\cos\phi(\nu)$$
$$-\{\sin[Nh(\mu)+Nh(\nu)]+\sin[Nh(\mu)-Nh(\nu)]\}\left(\frac{b}{\mu}+(-1)^N\frac{a}{\mu}\right)\sin\phi(\mu)\cos\phi(\mu)\Bigg]. \tag{5.7}$$

Squaring and averaging, we get after some tedious algebra

$$\langle K_N^2(\mu,\nu)\rangle=\frac{R_N}{4N^2(\mu-\nu)^2f(\mu)f(\nu)2\nu\mu(b^2-a^2)^2/4}\Bigg(2\nu\mu\frac{(b^2-a^2)^2}{2}-\nu^2\mu^2[2ab(-1)^N+a^2+b^2]$$
$$+\frac{(a^2+b^2)}{2}(\nu^2+\mu^2)[2ab(-1)^N+a^2+b^2]-2ab(-1)^Na^2b^2-(a^2+b^2)\frac{(b^4+a^4)}{2}$$
$$-(\nu^2+\mu^2)\frac{(b^2-a^2)^2}{2}+\frac{2}{4}(b^2-a^2)^2(a^2+b^2)\Bigg). \tag{5.8}$$

Simplifying for N even,

$$\langle K_N(\mu,\nu)^2\rangle=\frac{R_N(a+b)^2}{4N^2(\mu-\nu)^2f(\mu)f(\nu)\nu\mu\frac{(b^2-a^2)}{2}}\times[\nu\mu(b^2+a^2)-\nu^2\mu^2-a^2b^2+(\nu-\mu)^2ab] \tag{5.9}$$

while for N odd,

$$\langle K_N(\mu,\nu)^2\rangle=\frac{R_N(a-b)^2}{4N^2(\mu-\nu)^2f(\mu)f(\nu)\nu\mu\frac{(b^2-a^2)}{2}}\times[\nu\mu(b^2+a^2)-\nu^2\mu^2-a^2b^2-(\nu-\mu)^2ab]. \tag{5.10}$$

Note that $R_{N_{\text{even}}}(a+b)^2=A(a+b)^2=[(a-b)^2/4](a+b)^2=(a^2-b^2)^2/4$ and $R_{N_{\text{odd}}}(a-b)^2=B(a-b)^2=(a^2-b^2)^2/4)$. Comparing this expression with that found by the previous method of Sec. IV, we find that $C=(-1)^Nab$. Thus C is identified as a symmetry breaking parameter, which is a new concept missing from earlier treatments of this model. The standard large-N limit techniques of analyzing matrix models like the loop equation method Refs. [1,6] and renormalization group Ref. [4] assume a smooth behavior with respect to N at large N. The result that C differs for odd or even N by terms of order 1 suggests that these methods may need to be revisited in the context of random matrix models with eigenvalue distributions with gaps.

VI. CONCLUSION

To conclude, we have outlined a method which reproduces known results for the single-cut model and extended it to the two-cut random matrix model. The two-point density-density correlator contains a derivative part familiar from the single-cut model but in addition contains a nontrivial nonderivative piece. It is further seen that different methods give different values for the two-point correlator. The orthogonal polynomial method is briefly outlined and gives different values for the nonderivative piece for even and odd eigenvalues. The loop equation method gives a different result. The difference in the results is in the nonderivative part of the two-point density-density correlator. The method outlined unifies these differences in an arbitrary constant C (which cannot be fixed by the chemical potential constraint and is the symmetry breaking parameter) which takes different values. Different values of C found from the orthogonal polynomial and loop equation methods are identified.

This raises several questions regarding the analysis of this model. One possibility is that the even-odd differences may require some care in handling the large N techniques of random matrix models, e.g., loop equations and renormalization group. Another question relates to spontaneous breaking of the Z_2 symmetry in the large N limit. In this context, for the Z_2 symmetric random matrix models with two wells, an infinite family of solutions, which break the Z_2 symmetry and have the same free energy as the Z_2 symmetric solution but different connected correlators, has been identified in Ref. [8]. It would be interesting to compare whether the different solutions noted here (corresponding to different values of C) correspond to some of the multiple solutions (spontaneous symmetry breaking solutions) of Ref. [8]. Finally let us note that when the number of connected components for the support of the eigenvalues changes, one finds a new universality class for the correlators. It is thus not completely obvious that it is legitimate to use the simple one-cut function in the application to mesoscopic fluctuations. These new results for the double-well random matrix model which incorporate spontaneous symmetry breaking effects are found for all $\beta=1,2$, and 4. It seems interesting to us that this simple system, namely N charges confined by a symmetric doublewell with a logarithmic repulsion between the charges, exhibits such rich behavior.

Note added. In Ref. [7] among other things the even part of the two-point correlator Eq. (5.9) has been found. After this work was completed we became aware of Ref. [9], in which the odd part of the two-point correlator is also found by a method due to Shohat.

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APPENDIX A

Equation (4.15) was derived under the assumption that a counterpart of $(\delta\overline{\rho}/\delta a_i)_{V,b_i}=0$. Here we prove this result for the asymmetric double well. [Following a similar procedure we can show that an equivalent form of $(\delta\overline{\rho}/\delta a_i)_{V,b_i}=0$, from which Eq. (3.12) follows for the symmetric double well.] From Eq. (4.15) it is easy to see that we have to prove the following equation equivalent to $(\delta\overline{\rho}/\delta a_i)_{V,b_i}=0$ in the single-well problem:

$$\frac{\left(-2\pi\overline{\rho}(\lambda)\epsilon_\lambda\frac{\delta\sqrt{|\sigma(\lambda)|}}{\delta_a}+2\pi\overline{\rho}(\mu)\epsilon_\mu\frac{\delta\sqrt{|\sigma(\mu)|}}{\delta a}\right)}{(\lambda-\mu)} \tag{A1}$$

$$=\frac{1}{\pi\beta}\int_a^b\frac{V'(x)\frac{\delta\sqrt{|\sigma(x)|}}{\delta a}}{(x-\lambda)(x-\mu)}-\frac{1}{\pi\beta}\int_c^d\frac{V'(x)\frac{\delta\sqrt{|\sigma(x)|}}{\delta a}}{(x-\lambda)(x-\mu)}. \tag{A2}$$

Let us take

$$2G(z)=\frac{V'(z)}{\beta}-P(z)\sqrt{(z-a)(z-b)(z-c)(z-d)}. \tag{A3}$$

Multiply by $[1/(z-u)(z-v)][\sqrt{\sigma(z)/(z-a)}]$ and integrate over a large circle. The first term

$$2\oint_c\frac{G(z)}{(z-u)(z-v)}\frac{\sqrt{\sigma(z)}}{(z-a)}\frac{dz}{2\pi i}=\oint_c\frac{1}{z^2}dz=0 \tag{A4}$$

as for large z $G(z)\approx 1/z$, $\sqrt{\sigma(z)}\approx z^2$, $1/(z-u)(z-v)\approx 1/z^2$, and $1/z-a\approx 1/z$. The third term becomes

$$\oint_c\frac{P(z)\sigma(z)}{(z-u)(z-v)(z-a)}\frac{dz}{2\pi i}=\frac{P(u)(u-b)(u-c)(u-d)-P(v)(v-b)(v-c)(v-d)}{(u-v)} \tag{A5}$$

while the second term is

$$\oint_c\frac{V'(z)}{(z-u)(z-v)}\frac{\sqrt{\sigma(z)}}{(z-a)}\frac{dz}{2\pi i}$$

$$=\frac{\sqrt{\sigma(u)}V'(u)}{(u-a)(u-v)}+\frac{\sqrt{\sigma(v)}V'(v)}{(v-a)(v-u)}+\frac{1}{\pi i}\int_a^b\frac{V'(x)}{(x-u)(x-v)}\frac{\sqrt{\sigma(x)}}{(x-a)}+\frac{1}{\pi i}\int_c^d\frac{V'(x)}{(x-u)(x-v)}\frac{\sqrt{\sigma(x)}}{(x-a)}. \tag{A6}$$

On using $u=\lambda+i\epsilon$ and $v=\mu+i\epsilon$ with λ,μ on the right-hand cut, $1/x-\lambda-i\epsilon=(P/x-\lambda)+i\pi\delta(x-\lambda)$ and $\sqrt{\sigma(x)}=i\epsilon_\lambda|\sqrt{\sigma(x)}|$, the second integral simplifies to

$$\oint_c\frac{V'(z)}{(z-u)(z-v)}\frac{\sqrt{\sigma(x)}}{(z-a)}\frac{dz}{2\pi i}$$

$$=\frac{1}{\pi}\int_a^b\frac{V'(x)|\sqrt{\sigma(x)}|dx}{(x-\lambda)(x-\mu)(x-a)}-\frac{1}{\pi}\int_c^d\frac{V'(x)|\sqrt{\sigma(x)}|dx}{(x-\lambda)(x-\mu)(x-a)}. \tag{A7}$$

Combining these three terms and simplifying we get Eq. (A2), which is what is needed in order to get Eq. (4.15).

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